Mechanism Design with Approximate Valuations

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generate good outcomes for data you don't have

generate good outcomes for data you don't have by leveraging the players' **KNOWLEDGE** and **RATIONALITY**

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<u>classically</u>

YES, if player's self-knowledge is **EXACT**

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TODAY

YES, if player's self-knowledge is **APPROXIMATE** (in single-good auctions)

GOAL

give Rolex to player who values it the most (max. *social welfare*)







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use auction mechanism to extract information from the players





bidding true valuation is a (very weakly) dominant strategy



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WARNING!

optimal performance from an ASSUMPTION:



optimal performance from an ASSUMPTION:

each player knows his own valuation exactly





optimal performance from an ASSUMPTION:

each player knows his own valuation exactly





optimal performance from an **ASSUMPTION:**

each player knows his own valuation exactly



Either:

(a) it does not make any difference (b) exact knowledge is VERY strong assumption



optimal performance from an ASSUMPTION:

each player knows his own valuation exactly



WEAKER ASSUMPTION: Bayesian? each player knows his own individual Bayesian



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<u>same</u> second-price mechanism: just truthfully bid your expected value



WEAKER ASSUMPTION: Bayesian? each player knows his own individual Bayesian



Does player 2 really know that Pr(16k) = 1.5 Pr(16.6k)? If no, and it matters, still very strong!

NEED A MORE CONSERVATIVE MODEL

OUR APPROXIMATE KNOWLEDGE MODEL











approximate knowledge induces *inaccuracy param*. δ (above, all players within inaccuracy e.g. $\delta = 40\%$)



Q:

How well can we leverage approximate knowledge?

HOW TO MEASURE PERFORMANCE?

auction mechanism


























QUESTION (now more precise) What is the max $\varepsilon(\delta, n)$ that we can guarantee?

Our Results

How about dominant strategies?



How about dominant strategies?



Old Theorem

if $\delta = 0 \exists$ dominant-strategy mechanism guaranteeing $100\% \cdot MSW$

New Theorem

if $\delta > 0 \exists$ dominant-strategy mechanism guaranteeing $(1 - \delta) \cdot MSW$?

New Theorem

if $\delta > 0 \exists$ dominant-strategy mechanism guaranteeing $(1 - \delta)^2 \cdot MSW$?









Remark 1: dominant-strategy mechanism exist

Remark 2: they perform terribly!

a random assignment trivially guarantees $\frac{1}{n}$





Interpretation

dominant strategy useful iff exact knowledge or Bayesian





 $70(1 \pm 0.1)?$

Interpretation

dominant strategy useful iff exact knowledge or Bayesian













A New World

Dominant strategies not useful...

What other solution concepts could make sense? undominated strategies [Jackson, BLP]

















Thm 2: Second-price mechanism inundominated strats. guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$



(note that the second-price mechanism is not dominant-strategy anymore!)



(note that the second-price mechanism is not dominant-strategy anymore!)

\Rightarrow a new role for undominated strategies!



Thm 3: \forall <u>deterministic</u> undom. strat. mechanism guarantees no more than $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$

Harder!

dominant strategies \rightarrow ``single'' mechanism (rev. principle)

undominated strategies \rightarrow infinitely many mechanisms



Thm 3: \forall deterministic undom. strat. mechanism guarantees no more than $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$

And with randomness?

Implementation in Undomin. Strat's



Implementation in Undomin. Strat's





Summary

Dominant Strategies

Thm 1: Dominant Strategies don't work

Undominated Strategies

Thm 2: Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$

Thm 4: Our mechanism

guarantees $\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \cdot MSW$

Thm 3: & it is optimal among deterministic mechanisms

Thm 5: & it is optimal among probabilistic mechanisms

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Structural Theorems

understanding undominated strategies with approximate knowledge
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Lower Bound Tool: Distinguishable Monotonicity Lemma Thm 3: & it is optimal among deterministic mechanisms

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Undominated Intersection Lemma: $|K_i \cap K'_i| \ge 2 \Longrightarrow UDed_i(K_i) \cap UDed_i(K'_i) \neq \emptyset$

 $K_1 = K_2 = K_3 = [(1 - \delta)x, (1 + \delta)x]$



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 $K_1 = K_2 = K_3 = [(1 - \delta)x, (1 + \delta)x]$



Approximate Knowledge

more adversarial... ... more work (but doable) ... more fun!

Thank you!

Will use:

Undominated Intersection Lemma:

A tool for undominated strategy mechanisms

- No revelation principle to help
- Need to apply to all mechanisms

Undominated Intersection Lemma: $|K_i \cap K'_i| \ge 2 \Longrightarrow UDed_i(K_i) \cap UDed_i(K'_i) \neq \emptyset$

Example:









Undominated Intersection Lemma: $|K_i \cap K'_i| \ge 2 \Longrightarrow UDed_i(K_i) \cap UDed_i(K'_i) \neq \emptyset$

Recall Theorem 3:

in undominated strategies, no deterministic mechanism guarantees more than $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$

Proof:

1. Pick any x and set $K_1 = K_2 = K_3 = [(1 - \delta)x, (1 + \delta)x]$



- 1. Pick any x and set $K_1 = K_2 = K_3 = [(1 \delta)x, (1 + \delta)x]$
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- 3. Apply UIL to obtain $\sigma_i \in \text{UDed}_i(K_i) \cap \text{UDed}_i(K'_i)$ for each *i*



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- 4. When playing $(\sigma_1, \sigma_2, \sigma_3)$, someone is **unlucky**, WLOG player 1



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- 5. Choose the "world" of (K_1, K'_2, K'_3) ... This is the hard instance!



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Recall...



Recall...







n



1. Motivation


















Today's focus



Today's focus



Today's focus



Within 1% or 10% or 25% ...

Approximate Types





Approximate Types







 $TV_i \coloneqq$ "true valuation for player *i*" $K_i \coloneqq$ "approximate valuation for player *i*"











constructible plot of land































Approximate Valuations (summary)



Approximate Valuations (summary)



approximation inaccuracy δ is:

guaranteed



3. Our Question

Which Social Welfare?





Which Social Welfare?







 $\delta = 0$ $\Rightarrow \text{ guaranteed } SW \text{ is maximum}$

(by the second-price mechanism)





 \Rightarrow guaranteed SW decreases





 δ increases <u>further</u> \Rightarrow guaranteed *SW* decreases <u>further</u>








What is the best $\varepsilon(\delta, n)$?

Under which solution concepts should we ask the question?



What is the best $\varepsilon(\delta, n)$?

Under which solution concepts should we ask the question?

(non-Bayesian) incomplete information, so two natural notions to consider:

- 1. implementation in **dominant** strategies
- 2. implementation in **undominated** strategies



What is the best $\varepsilon(\delta, n)$?

Under which solution concepts should we ask the question?

(non-Bayesian) incomplete information, so two natural notions to consider:

- 1. implementation in **dominant** strategies
- 2. implementation in **undominated** strategies
- (3. ex-post NE reduces to dominant strategies)

Implementation in ...

Implementation in ...



Implementation in ...



Implementation in ...



Implementation in ...

... Dominant Strategies

irrelevant!?

Implementation in ...

... Dominant Strategies

irrelevant!?

maybe not ... as there is more to reveal

Implementation in ...

... Dominant Strategies

• Thm 1: Can guarantee $\varepsilon(\delta, n) \cdot MSW$

Implementation in ...

... Dominant Strategies

Thm 1:
Can guarantee (?) · MSW

Implementation in ...

... Dominant Strategies

• Thm 1: Can guarantee $(1 - \delta) \cdot MSW$

Implementation in ...

• Thm 1:
Can guarantee
$$\left(\frac{1-\delta}{1+\delta}\right) \cdot MSW$$
?

Implementation in ...

• Thm 1:
Can guarantee
$$\frac{(1-\delta)^5}{(1+\delta)^3}$$
 MSW?

Implementation in ...

... Dominant Strategies

Implementation in ...

... Dominant Strategies

Thm 1:
 Cannot get more than ¹/_n · MSW
 Terrible!
 can trivially achieve by assigning good <u>at random</u> (after all, <u>some</u> player has the highest valuation!)

Implementation in ...

... Dominant Strategies

- Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$
 - Terrible!
 - can trivially achieve by assigning good <u>at random</u> (after all, <u>some</u> player has the highest valuation!)

Interpretation:

dominant strategy if and only if exact knowledge of valuation or individual Bayesian

Implementation in ...



Implementation in ...



Implementation in ...

- Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$ Terrible! can trivially achieve by assigning good at random (after all, some player has the highest valuation!) $70(1 \pm 0.1)$ **Interpretation:** $70(1 \pm 0.01)$ dominant strategy $70(1 \pm 0.001)$ if and only if
 - exact knowledge of valuation or individual Bayesian

Implementation in ...

... Dominant Strategies

• Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$ Terrible! can trivially achieve by assigning good <u>at random</u> (after all, <u>some</u> player has the highest valuation!) 70(1 \pm 0.1) 70(1 \pm 0.01) Interpretation: dominant strategy



Implementation in ...



Implementation in ...



Implementation in ...

... Dominant Strategies

... Undominated Strategies

Implementation in ...

... Dominant Strategies

... Undominated Strategies



Implementation in ...

... Dominant Strategies

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Implementation in ...

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Implementation in ...

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Implementation in ...

... Dominant Strategies

• Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

... Undominated Strategies

Thm 2:

Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$
Implementation in ...

... Dominant Strategies

Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

... Undominated Strategies

Thm 2:



(note that the second-price mechanism is not dominant-strategy anymore!)

Implementation in ...

... Dominant Strategies

Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$ Second-price mechanism $(1-\delta)^2$

... Undominated Strategies

Thm 2:



(note that the second-price mechanism is not dominant-strategy anymore!)

Implementation in ...

... Dominant Strategies

• Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

... Undominated Strategies

Thm 2:

- Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$
- And it is optimal among deterministic mechanisms

Implementation in ...

... Dominant Strategies

Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

... Undominated Strategies

Thm 2:

- Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$
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Solution ● Thm 3:

• Our mechanism guarantees $\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \cdot MSW$

Implementation in ...

... Dominant Strategies

• Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

... Undominated Strategies

Thm 2:

- Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$
- And it is optimal among deterministic mechanisms

● Thm 3:

even better!!!

$$(1-\delta)^2 + \frac{4\delta}{n} \cdot MSW$$

Implementation in ...

... Dominant Strategies

Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

... Undominated Strategies

Thm 2:

- Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$
- And it is optimal among deterministic mechanisms

● Thm 3:

• Our mechanism guarantees
•
$$\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \cdot MSW$$

• And it is optimal among
probabilistic mechanisms

Implementation in ...

... Dominant Strategies

Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

$$\delta = 0.5 \quad n = 2 \quad 5 \text{ times better}$$

$$\delta = 0.5 \quad n = 4 \quad 3 \text{ times better}$$

$$\delta = 0.25 \quad n = 2 \quad 2 \text{ times better}$$

... Undominated Strategies

Thm 2:

- Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$
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Thm 3:

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Implementation in ...

... Dominant Strategies

Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

... Undominated Strategies

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Implementation in ...

... Dominant Strategies

• Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$

Upper Bound Tool: Undominated Intersection Lemma

... Undominated Strategies

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Implementation in ...

... Dominant Strategies

... Undominated Strategies



Implementation in ...

... Dominant Strategies

... Undominated Strategies



Full version at http://arxiv.org/abs/1112.1147

5. Our Techniques

Implementation in ...

... Dominant Strategies



Lower Bound Tool: Distinguishable Monotonicity Lemma

... Undominated Strategies

Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$ And it is optimal among deterministic mechanisms Thm 3: Our mechanism guarantees

$$\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \cdot MSW$$

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A tool for undominated strategy mechanisms

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Tool #1

Undominated Intersection Lemma: $|K_i \cap K'_i| \ge 2 \Longrightarrow UDed_i(K_i) \cap UDed_i(K'_i) \neq \emptyset$





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Implementation in ...

... Undominated Strategies

... Dominant Strategies



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Tool #2

Approximate truthfulness:

 $UDed_i(K_i) \subset [\min K_i, \max K_i]$



Tool #2

Distinguishable Monotonicity Lemma:

For any mechanism satisfying good property: $UDed_i(K_i) \subset [\min K_i, \max K_i]$



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What property?

Tool #2

Distinguishable Monotonicity Lemma:

For any mechanism satisfying good property: $UDed_i(K_i) \subset [\min K_i, \max K_i]$

Recall a classical result:
 \forall monotonic $f: \mathbb{R}^n \rightarrow [0,1]^n$, M_f is DST

Tool #2

Distinguishable Monotonicity Lemma:

For any mechanism satisfying good property: $UDed_i(K_i) \subset [\min K_i, \max K_i]$

- Recall a classical result:
 \forall monotonic $f: \mathbb{R}^n \rightarrow [0,1]^n$, M_f is DST
- How does M_f look like? On input bid-profile v
 - Player *i* wins w.p. $f_i(v)$;
 - Player *i* (if wins), pays $v_i \frac{1}{f_i(v)} \int_{z=0}^{v_i} f_i(z \sqcup v_{-i}) dz$

Tool #2

Distinguishable Monotonicity Lemma:

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Our result:
Lower Bound Tool

Tool #2

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- Our result:
 - * = distinguishably monotonic

Lower Bound Tool

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- Our result:
 - * = distinguishably monotonic

 \Rightarrow only need to focus on finding good allocation function f

Our Results

Implementation in ...

... Dominant Strategies

Distinguishable Monotonicity Lemma

• Thm 1: Cannot get more than $\frac{1}{n} \cdot MSW$ • Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$ • And it is optimal among deterministic mechanisms • Thm 3: • Our mechanism guarantees $\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \cdot MSW$

> And it is optimal among probabilistic mechanisms

... Undominated Strategies



Upper Bound Tool

Undominated Intersection Lemma: $|K_i \cap K'_i| \ge 2 \Longrightarrow UDed_i(K_i) \cap UDed_i(K'_i) \neq \emptyset$

• Thm: in undominated strategies, no deterministic mechanism guarantees more than $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$.

Proof:

1. Pick any x and set $K_1 = K_2 = K_3 = [(1 - \delta)x, (1 + \delta)x]$



- 1. Pick any x and set $K_1 = K_2 = K_3 = [(1 \delta)x, (1 + \delta)x]$
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- 3. Apply UIL to obtain $\sigma_i \in \text{UDed}_i(K_i) \cap \text{UDed}_i(K'_i)$ for each *i*



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6. Conclusion

Conclusion



mechanism design =

Conclusion



mechanism design =











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And it is optimal among probabilistic mechanisms



There is a profile of strategies of players that "cannot be beaten" for which the mechanism M ensures good social welfare.

Classical Model

•
$$\sigma_i \geq_{TV_i}^{vw} \sigma'_i$$
 if $\forall \tau_{-i}, \ u_i (TV_i, M(\sigma_i \sqcup \tau_{-i})) \ge u_i (TV_i, M(\sigma_i' \sqcup \tau_{-i}))$

•
$$\operatorname{Dnt}_{i}(\operatorname{TV}_{i}) = \left\{ \sigma_{i} : \forall \sigma_{i}', \sigma_{i} \geq TV_{i} \\ TV_{i}'' \right\} =$$
"unbeatable strategies w.r.t. TV_{i} "

A mechanism M implements $\varepsilon \cdot MSW$ if

 $\forall TV \quad , \ \exists \sigma \in \text{Dnt}(TV)$

$$\mathbb{E}_{M,\sigma}\left[SW(TV, M(\sigma))\right] \ge \varepsilon \cdot MSW(TV)$$

There is a profile of strategies of players that "cannot be beaten" for which the mechanism M ensures good social welfare.

Our Model

$$\sigma_{i} \geq_{\substack{V \\ V \\ K_{i}}}^{VW} \sigma_{i}' \text{ if } \forall TV_{i} \in K_{i}, \forall \tau_{-i}, \ u_{i} (TV_{i}, M(\sigma_{i} \sqcup \tau_{-i})) \geq u_{i} (TV_{i}, M(\sigma_{i}' \sqcup \tau_{-i}))$$

$$\text{Dnt}_{i}(\mathsf{TV}_{i}) = \left\{ \sigma_{i} : \forall \sigma_{i}', \sigma_{i} \geq \mathsf{VW}_{i} \\ \mathsf{TV}_{i} \\ \mathsf{K}_{i} \\ \mathsf{K}_{i}$$

A mechanism $M \delta$ -implements $\varepsilon \cdot MSW$ if

 $\begin{array}{l} \forall K \in [\delta], \forall TV \in K, \ \exists \sigma \in \operatorname{Dnt}(\mathcal{T}_{K}) \\ & \underset{M,\sigma}{\mathbb{E}} \left[SW(TV, M(\sigma)) \right] \geq \varepsilon \cdot MSW(TV) \end{array}$

Thm: in dominant strategies, every (possibly probabilistic) mechanism cannot guarantee more than $\frac{1}{n} \cdot MSW$

"Revelation Principle" *

Claim: in dominant strategies, every (possibly probabilistic) <u>direct</u> mechanism cannot guarantee more than $\frac{1}{n} \cdot MSW$ set of strategies is $[\delta]^*$

- Claim: in dominant strategies, every (possibly probabilistic) <u>direct</u> mechanism cannot guarantee more than $\frac{1}{n} \cdot MSW$
- Proof:
 - 1. Lemma: a dominant strategy direct mechanism M gives the same outcome when a player deviates individually: $\forall K, \forall i, \forall K'_i$,

 $M_i(K) = M_i(K'_i \sqcup K_{-i})$ (Proof: play with different worlds)

- 2. Consider WORLD₁: all players bid low; there is some unlucky player, say player 5.
- 3. Consider WORLD₂: all players bid low except player 5.
- 4. Compute and conclude. QED



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"When players use non-stupid strategies, the mechanism M ensures good social welfare"

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Classical Model



 $\forall \tau_{-i}, \ u_i (TV_i, M(\sigma_i \sqcup \tau_{-i})) \le u_i (TV_i, M(\sigma_i' \sqcup \tau_{-i}))$ $\exists \tau_{-i}, \ u_i (TV_i, M(\sigma_i \sqcup \tau_{-i})) < u_i (TV_i, M(\sigma_i' \sqcup \tau_{-i}))$

"When players use non-stupid strategies, the mechanism M ensures good social welfare"



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 $\begin{array}{l} \hline \textbf{Classical Model} \\ \bullet & \sigma_{i} \leq \sigma_{i}' \text{ if } \\ \hline \textbf{V}_{TV_{i}} \sigma_{i}' \text{ if } \\ \hline \textbf{V}_{T-i}, \ u_{i} (TV_{i}, M(\sigma_{i} \sqcup \tau_{-i})) \leq u_{i} (TV_{i}, M(\sigma_{i}' \sqcup \tau_{-i})) \\ \hline \textbf{V}_{T-i}, \ u_{i} (TV_{i}, M(\sigma_{i} \sqcup \tau_{-i})) < u_{i} (TV_{i}, M(\sigma_{i}' \sqcup \tau_{-i})) \\ \hline \textbf{V}_{better at least once} \\ \hline \textbf{VDed}_{i} (TV_{i}) = \\ \hline \textbf{V}_{i}: \ \nexists \sigma_{i}' \ s. t. \sigma_{i} \leq \sigma_{i}' \\ \hline \textbf{V}_{TV_{i}} \sigma_{i}' \\ \hline \textbf{V}_{i}: \ \textbf{V}_{i} = \left\{ \sigma_{i}: \ \nexists \sigma_{i}' \ s. t. \sigma_{i} \leq \sigma_{i}' \\ \hline \textbf{V}_{i}: \ \textbf{V}_{i} \\ \hline \textbf{V}_{i}: \ \textbf{V}_{i}: \ \textbf{V}_{i} \\ \hline \textbf{V}_{i}: \ \textbf{V}_{i}:$

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- 1.
- 2.

- 1.
- 2. Done!

- 1. $UDed_i(K_i) \subset [\min K_i, \max K_i]$ for every player *i*
- 2. Done!

Thm: Second-price mechanism guarantees $\left(\frac{1-\delta}{1+\delta}\right)^2 \cdot MSW$

Proof:

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Why are we done?

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Why are we done? Hardest instance is still:



Second-price: QED

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For any τ_{-i} , have three cases: *i* loses in $\sigma_i \sqcup \tau_{-i}$; *i* wins in $\sigma'_i \sqcup \tau_{-i}$ *i* loses in $\sigma_i \sqcup \tau_{-i}$; *i* loses in $\sigma'_i \sqcup \tau_{-i}$ *i* wins in $\sigma_i \sqcup \tau_{-i}$; *i* wins in $\sigma'_i \sqcup \tau_{-i}$ *i* wins in $\sigma_i \sqcup \tau_{-i}$; *i* loses in $\sigma'_i \sqcup \tau_{-i}$

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only interesting one-

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Done! 2.



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Utility of player *i*:

 $\begin{array}{l} \sigma_i' \sqcup \tau_{-i} < 0 \\ \sigma_i \sqcup \tau_{-i} = 0 \end{array} \ \, \text{No matter what the} \\ \text{devil chooses} \end{array}$

 $UDed_i(K_i)$ K_i

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Task reduces to designing a good f

Lower Bound Tool

Distinguishable Monotonicity Lemma: $\forall \text{ monotonic}^* f : \mathbb{R}^n \rightarrow [0,1]^n, \quad M_f \text{ satisfies}$ $UDed_i(K_i) \subset [\min K_i, \max K_i]$

- Designing f:
 - When bids are close to each other: give good at random.
 - When there is a "clear winner": act like second-price.
 - If neither: interpolate in a smart way.

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- Designing f:
 - When bids are close to each other: give good at random.
 - When there is a "clear winner": act like second-price.
 - If neither: interpolate in a smart way.

This is delicate. In every "intermediate case", need to:

- 1) ensure the target social welfare, and
- 2) ensure distinguishable monotonicity.



<u>"10000m" view:</u>

1. On input bids (v_1, v_2, \dots, v_n) , WLOG $v_1 \ge v_2 \ge \dots \ge v_n$.

"10000m" view:

- $\begin{array}{c|c} \underline{\text{D000m'' view:}} & & 1, 2, 3, ..., n^* \\ 1. & \text{On input bids } (v_1, v_2, ..., v_n), \text{WLOG } v_1 \geq v_2 \geq \cdots \geq v_n. \end{array}$
- 2. Find "magic" threshold n^*



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Assign good to only "candidate winning" players 1,2, ..., n^* 3. where player $i \in \{1, 2, ..., n^*\}$ wins with "magic" probability:



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2. Find "magic" threshold
$$n^*$$
 s.t.
$$\begin{cases} v_i > \frac{\sum_{j=1}^{n^*} v_j}{n^* + D(\delta)} & \text{for all } 1 \le i \le n^* \\ v_i \le \frac{\sum_{j=1}^{n^*} v_j}{n^* + D(\delta)} & \text{for all } n^* < i \le n \end{cases}$$

Assign good to only "candidate winning" players 1,2, ..., n^* 3. where player $i \in \{1, 2, ..., n^*\}$ wins with "magic" probability:

$$f_i(v) = \frac{1}{n} \cdot \frac{n + D(\delta)}{n^* + D(\delta)} \cdot \frac{v_i(n^* + D(\delta)) - \sum_{j=1}^{n^*} v_j}{v_i D(\delta)}$$

Om" view:candidate winners
1, 2, 3, ..., n^* losers
 $n^* + 1, ..., n$ 1. On input bids $(v_1, v_2, ..., v_n)$, WLOG $v_1 \ge v_2 \ge \cdots \ge v_n$. "100m" view:

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Easy to evaluate, just like the second-price mechanism!

"1m" view:

animpted with a total probability mass of 1, by averaging three exists none place j such that $P_{0}^{(1)}(\ldots,c_{i}^{(1)}) \leq \frac{1}{2}$, i.e., place preserves the good with as more than $\frac{1}{2}$ redshifty. Without loss $D_{0}^{(1)}(\ldots,c_{i}^{(2)}) \leq \frac{1}{2}$. According to the most of the balance distribution of the second state of the second stat $i^{\alpha}(s_1 \cup s_{-1}^{\prime}) = 0$. Now consider a "world" with δ -approximate-valuation profile K and true-valuation profile tv as

 $K \stackrel{\mathrm{def}}{=} (\mathrm{int}_{\delta}(x), \mathrm{int}_{\delta}(y), \ldots, \mathrm{int}_{\delta}(y))$ and $\operatorname{tr} \stackrel{\mathrm{def}}{=} \left(\lfloor (1+\delta) x \rfloor, \lceil (1-\delta) y \rceil, \dots, \lceil ((1-\delta) y \rceil) \right)$ We have just shown that there exist some $s_1 \cup s_{-1}' \in UDad(K)$ satisfying $F_1^A(s_1 \cup s_{-1}') = 0$.

 $SW(te, F(s_1 \cup s'_{-1})) = |(1 - \delta)g| \le (1 - \delta)g + 1 \le \frac{x(1 - \delta)^2}{1 + x} + 3$ $\leq \frac{(1-\delta)^2}{(1+\delta)^2} |(1+\delta)x| + 4 \leq \left(\frac{(1-\delta)^2}{(1+\delta)^2} + \frac{4}{B}\right) |(1+\delta)x|$ $= \left(\frac{(1-\delta)^2}{(1+\delta)^2} + \frac{4}{B} \right) MSW(nr) .$ and thus the claimed inequality hods.

E Proof for Theorem 3

E.1 Proof of Statement 1

In 1 results of sourcements in the solution of restrict superbrase to consider only f-mechanism (result Solution 3.2), as no between or binding-induced bornearing frames (format B d) and (f) and (f

Step 1

Step 1 we start by partiag forward s set of constraints for the winning-pointality function), and prove the they are utilized for ensuing the isage function of the maximum acids within a summing (for which the set of a stabilized matricipe consists with the start of all possible stabilized possible. For easy $\delta \in (0,1)$, doin $m_{1}^{-1} \frac{1}{2} \left[\frac{1}{2} \right]^{2} - 1 = \frac{1}{2} \frac{1}{m_{1}^{2}} > 0$. We asy that a similar possibility function $\left[i \in J = 2m_{1}^{2} i \in J = 1 \right]$ multiply maximum of

 $\forall i \in N, \forall v \in \{0, 1, ..., B\}^{N}, \sum_{i=1}^{2} f_{j}(v)v_{j} + D_{k} \cdot f_{i}(v)v_{i} \ge \frac{1}{n} \cdot v_{i}(v + D_{k})$. (R.1) We prove the following lemma:

The second inequality of this last equation yields a contradiction with the first inequality in the arms sensition $\frac{S^{\pm}}{n^{+}} > \frac{S^{\pm} + S^{\pm}}{n^{+} + n^{+} + D_{4}} \leftrightarrow \frac{S^{\pm}}{n^{+}} > \frac{S^{\pm}}{(n^{+} + D_{4})} \ .$

(1) Substituting Equation E.8 into the definition of p^(B) (defined in Equation E.7) we immediately have p^(B)(z) ≥ 0 for each player i. Summing the p^(B)_i ap over all the players i we get:

 $\sum_{i=1}^{0} f_{i}^{(0)}(z) = \frac{1}{n} \cdot \frac{n+D_{k}}{n^{*}+D_{k}} \cdot \sum_{i=1}^{n^{*}} \frac{z_{i}(n^{*}+D_{k}) - \sum_{j=1}^{n^{*}} z_{j}}{z_{i}D_{k}}$ $= \frac{1}{n} \cdot \frac{n + D_k}{(n^* + D_k)D_k} \cdot \left(n^*(n^* + D_k) - \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} \frac{z_j}{z_i}\right)$

 $\leq \frac{1}{n} \cdot \frac{n + D_4}{(n^* + D_4)D_4} \cdot (n^*(n^* + D_4) - n^*n^*) = \frac{n + D_4}{n} \cdot \frac{n^*}{n^* + D_4} \leq 1 \ .$ In particular, $f_1^{(0)}(z) \leq 1$ for each player i, as we have already established that $f_2^{(0)}(z)$ in mon-negative. And thus, after symmetrically extending the above argument to all other z, we deduce that $f^{(0)}$ is a valid winning-probability function.

(b) For notational simplicity assume that i = n (so that $z_{-1} = z_{-1}$) and also assume $z_2 \ge z_2 \ge \cdots \ge z_{n-1}$. (As usual, the other cases follow by symmetry by appropriate re-infeding.)

So define n' to be number of Winners when only considering the first (n-1) values, i.e., when only considering $z_{0,\dots,n}, z_{n-1}$. We claim that:

 $s_k \leq \frac{\sum_{i=1}^{d} s_i}{d + D_k} \longrightarrow f_0^{(2)}(s) = 0 \text{ (i.e., s is a loser)}$ (E.9) $z_6 > \frac{\sum_{i=1}^{d} z_i}{z_i^2 + D_4} \longrightarrow f_n^{(2)}(z) > 0 \text{ (i.e., κ is a winner)}$ (E.10)

The implication of Equation E.9 is dear, because the number of winners when places a is present is still u'. We now argue that the implication of Equation E 10 also holds which is han of dealows So assume by way of contradiction that player n is a loser and yet $z_n > \frac{V_n^{(1)} + V_n}{2 + 2 N_n}$. Let the current number of winners be n^{*} (i.e., when player n is present), so we know that $v_n \leq \frac{\sum_{i=1}^{n} a_i}{a + p_i}$; in particular, we must have that $u' \neq u^*$ (otherwise we are done, as there is already a contradiction). However, for this choice of n^* , we have that:

 $\forall i \in \{n^* + 1, ..., n - 1\}, \quad z_i \leq \frac{\sum_{j=1}^{n^*} z_j}{n^* + D_2}$, But, this means that both n' and n' are the number of winners when player n is absent, contradiction the universes from Iten 3. Thus, Econtrice E.10 also holds.

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Lemma E.1. If f is degoed, then the vectorison $M_f = (\Sigma, F_f)$ parameters by ionology the Dirito graduatic Monotonicity Lemma: (Lemma (Lemma E.4) with f has the following serial influer quantum is indeministed statistics: for every h-approximate voltation profile $K = (K_{11}, ..., K_{2n})$, every to π K and every $\pi \in Obs(K_1)$:

$\mathbb{E}\left[SW(tv, F_f(v))\right] \ge \left(\frac{(1-\delta)^2 + \frac{\delta^2}{4}}{(1+\delta)^2}\right) MSW(tv)$

Proof. For every player $i \in N$, let $x_i \in \mathbb{R}$ be such that $K_i \subset int_i(x_i)$, and let $int_i(x) - int_i(x_i)$.

 $\sum_{i=1}^{n} \operatorname{tr}_{I} f_{I}(v) = \operatorname{SW}(\operatorname{tr}, F_{I}(v)) \geq \left(\frac{(1-\delta)^{2} + \frac{\delta}{2}}{(1+\delta)^{2}} \right) \operatorname{tr}_{I} \, .$ (E.2)

 $\begin{array}{l} \sum_{i=1}^{j \neq i} (i + i + i - j) \\ Alleo, ince every for a large <math display="inline">c \in S^*$ by Lemmas B-A, we must have $(1 - \delta)x_i \leq v_i \leq \max K_i \leq (1 + \delta)x_i, \\ \delta |x_i, \operatorname{sime} f + i + |\partial M| \ and \ K_i \subset \operatorname{int}_{\delta}(x_i). \\ Moreover, \ iv \in K \ indicater \ that, \ (1 - \delta)x_i \leq \Theta_i \leq (1 + \delta)x_i, \\ \operatorname{conthining} \text{these two we have} \end{array}$

 $\forall i \in \mathbb{N}, \forall v \in \{0, 1, \dots, B\}^{R}, \quad \begin{cases} \sum_{j=1}^{m} (\frac{1-2j}{1+2j})v_j f_j(t) \geq \left(\frac{(1-2j-1)!}{(1+2j)}\right) (\frac{1-2j}{1+2j})v_j f_j(v) > \left(\frac{2j-2j+2!}{(1+2j)}\right) (\frac{1-2j}{1+2j})v_j f_j(v) > \left(\frac{2j-2j+2!}{(1+2j)}\right) (\frac{1+2j}{1+2j})v_j f_j(v) > \left(\frac{2j-2j+2!}{(1+2j)}\right) (\frac{1+2j}{1+2j})v_j f_j(v) > (\frac{2j-2j}{1+2j}) (\frac{2j-2j}{1+2j})$

 $|\psi_{I_1} \psi_{I_2}| = \begin{cases} \sum_{j=0}^{n} \psi_j f_j(v) \ge \left(\frac{(1-\delta)^2 + \delta^2}{(1-\delta)^2}\right) v_i = \frac{4 + D_F}{n} \frac{1}{D^{-1}} - v_i , \\ \sum_{j=0}^{n} \psi_j f_j(v) = D_F v_i f_i(v) \ge \left((1-\delta)^2 + \frac{\delta^2}{n}\right) - \frac{1}{(1-\delta)^2} - v_i = \frac{\pi i D_F}{n} u_i \end{cases}$ (E.4) Note that Equation E.4 is the inequality required by the statement of the lemma, the other in-equality, Equation E.4, we show is implied by Equation E.5. Indiced.

 $\sum_{n=1}^{n} v_2 f_2(v) = \frac{1}{1 + D_2} \left(\sum_{n=1}^{n} v_2 f_2(v) + D_2 v_1 f_1(v) \right) \ge \frac{1}{1 + D_2} \frac{n + D_2}{n} v_1 \ .$ as desired. In sum, we have preved that as long as Equation E.5 is estimized for all $i \in N$ and for any $v \in \{0, 1, ..., R\}^R$, the social writer guarantees of Equation E.5 hold. 21

effore, we only need to prove that given two $z_m^{\pm}, z_m^{\pm} \in [0, B]^N$ such that $z_m^{\pm} > z_m^{\pm} >$ $\frac{\sum_{i=1}^{n} |z_i|}{2 + |z_i|} \text{ we have that } f_i^{(0)}(s_{i,n} \cup s_n^{(i)}) \geq f_i^{(0)}(s_{i,n} \cup s_n^{(i)}). Assume that when bicking a part of the set of t$ Assume by way of contradiction that $u^{\perp} < u^{\top} = u^{\perp} + u^{\pm}$. As before, let $S^{\perp} = \sum_{i=1}^{n^{\perp}} z_i$ and $S^{\rm T} = \sum_{i=1}^{N^{\rm T}} v_i - S^{\pm} + S^{\pm}.$ For every player $n^{\pm} \leq i < n$ $\frac{S^{\perp} + z_{0}^{\perp}}{s^{\perp} + 1 + D_{0}} < z_{0} \leq \frac{S^{\top} + z_{0}^{\top}}{s^{\top} + 1 + D_{0}} - \frac{S^{\perp} + S^{\perp} + z_{0}^{\top}}{s^{\perp} + s^{\perp} + 1 + D_{0}}$ Averaging over all $n^{\perp} \leq i < n^{\top}$ we get: $\frac{S^{\pm} + z_{0}^{\pm}}{u^{\pm} + 1 + D_{2}} < \frac{S^{\pm}}{u^{\pm}} \leq \frac{S^{\pm} + S^{\pm} + z_{0}^{\mp}}{u^{\pm} + u^{\pm} + 1 + D_{2}}$ but this is already a contradiction: $\frac{S^n}{n^n} \leq \frac{S^n + S^n + z_n^{\dagger}}{n^n + n^n + 1 + D_S} \Leftrightarrow \frac{S^n}{n^n} \leq \frac{S^n + z_n^{\dagger}}{n^n + 1 + D_S}$ $= \frac{S^{\pm}}{n^{\pm}} \le \frac{S^{\pm} + z_n^{\pm}}{n^{\pm} + 1 + D_1}$ and thus Equation E.11 holds. nor some Spatient is i i 10000. Nor we show entablished that $u^2 = u^2$. If $u^2 = u^2$ that $\int_0^{R_0} (x_{-u} \cup u_u^2) \geq \int_0^{R_0} (x_{-u} \cup u_u^2)$ $(x_0) = i$ introducing implied because they are using the same formula, and send that Equation 2.1 is an intervaling with respect to x_1 . If $u^2 > u^2$, i.i.th $u^2 = u^2 + u^2$, $u^2 = u^2$, $u^2 = u^2 = u^2 + u^2$, $u^2 = u^2 + u^2 + u^2 + u^2 + u^2$. If $u^2 > u^2 = u^2 + u^2$. $\frac{S^4}{u^4} > \frac{S^4 + z_a^2}{u^4 + 1 + D_4} = \frac{S^7 + S^4 + z_a^4}{u^7 + u^4 + 1 + D_4} \Rightarrow \frac{S^4}{u^6} > \frac{S^7 + z_a^4}{u^7 + 1 + D_4}$ (E.12) Using that, we do the calculating:

$$\begin{split} & f_0^{(0)}(x_{-6}+z_0^{-1}) - f_0^{(0)}(x_{-6}+z_0^{-1}) \\ & = C_1 ~ \Big(\frac{z_0^{-1}(w^++1+D_2) - S^{'}-z_0^{-1}}{(w^++1+D_2)z_0^{-1}} - \frac{z_0^{-1}(w^++1+D_2) - S^{*}-z_0^{-2}}{(w^++1+D_2)z_0^{-1}} \Big) \\ & = C_1 ~ \Big(\frac{S^{'}}{(w^++1+D_2)z_0^{-1}} - \frac{S^{''}+z_0^{-1}}{(w^++1+D_2)z_0^{-1}} \Big) \\ & = C_1 ~ \Big(\frac{S^{''}}{(w^++1+D_2)z_0^{-1}} - \frac{S^{''}+z_0^{-1}}{(w^++1+D_2)z_0^{-1}} \Big) \end{split}$$
 $= C_2 \cdot \left(\left(S^+ + z_n^+ \right) (a^+ + 1 + D_2) z_n^+ - \left(S^+ + z_n^+ \right) (a^+ + 1 + D_2) z_n^+ \right)$ $=C_2^{-1}\left((S^{+}+S^{+}+z^{+}_{u})(u^{+}+1+D_{\delta})z^{+}_{u}-(S^{+}+z^{+}_{u})(u^{+}+u^{+}+1+D_{\delta})z^{+}_{u}\right)$ $= D_2 \cdot \left(S^\top (n^\top + 1 + D_d) (z_n^\top - z_n^\perp) + S^\odot (u^\top + 1 + D_d) z_n^\top - u^\odot (S^\top + z_n^\top) z_n^\perp \right)$ $\geq C_{2^{-1}}\left(S^{\top}(u^{\top}+1+D_{\theta})(z_{u}^{2}-z_{u}^{1})+S^{u}(u^{\top}+1+D_{\theta})z_{u}^{2}-u^{u}(S^{\top}+z_{u}^{u})z_{u}^{2}\right)\geq 0$

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See 24 Sec 9 and the d analysis describes the straining periodicity function $f_{\rm c}$ being the Distinguished for the straining of the distinguished for the straining that an archivant will like the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be straining with the straining will be straining which will be straining with the straining will be straining with the strain

Definition E.2. For every $\delta \in (0, 1)$, and let $D_{\delta} \stackrel{\text{def}}{=} \left(\frac{|\delta|^2}{|\delta|^2} - 1 > 0$. We define the function $\rho(h; |\delta|, B^{(N)} \rightarrow 0, 1|^N)$ are follows:

• for every $z=(z_1,\ldots,z_k)\in[0,B]^N$ such that $z_k\ge z_2\ge\cdots\ge z_k,$ let u^* be the laset index in N such that $\forall i > n^*$, $t_i \le \frac{\sum_{j=1}^{n^*} z_j}{n^* + D_i}$,

call physers $1, \dots, n^n$ the winners and physers $n^n + 1, \dots, n$ the losers, and then set

 $f_{i}^{(0)}(s) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \frac{1}{n} \cdot \frac{n+D_{i}}{n+D_{i}} \cdot \frac{n(n^{*}+D_{i}) - \sum_{i=1}^{n} z_{i}}{nD_{i}}, & \text{if } i \leq n^{*}, \\ 0, & \text{if } i > n^{*}. \end{array} \right.$ (R.7)

• for other v_i , define $f^{(0)}$ by refeating it operators for all p_i , specifically, lattings to any permutation more that players such that $\pi(z) = (z_{i(1)}, \ldots, z_{i(n)})$ is non-introduce, we define $f_i^{(0)}(z) \stackrel{\text{def}}{\longrightarrow} f_{i(n)}^{(0)}(z(z))$.

 $L^{(0)}_{abc}(m)$ The detection of P^0 seems queue complexed, but the univerying matrixs is not that checks. The other detection of the second se

Here we have used the fact of $a_n^2-a_n^2>0$ and $S^0(a^2+1+B_0)-a^0(S^2+a_n^2)>0$ (by Equation E.12). This fields the proof rationing that $f^{(0)}$ is monotonic. The interpolating of $f^{(0)}$ is closure, because $f^{(0)}$ in closures constants, and there are at most a pixcae, as the number of winners decreases when a_n increases (recall Equation E.11).

(6) Fix a player i ∈ N and two distinct valuations v₀, v'₀ ∈ (0, 1, ..., B), and assume that v₀ < v'₀. We have already satablished the monotonicity and intergrability of f^(D), so that, to prove that f^(D) is i-DM, we only need to find a specific v₁, to make the integral positive (read)

 $\begin{array}{l} f(\eta \cup v_{-1}) = \frac{1}{n} \text{ since these are } n \text{ winners, all bidding the same valuation,} \\ f(v \cup v_{-1}) = \frac{1}{n \log} (D_0 + n - 1 - \frac{n}{n} (n - 1)) > \frac{1}{n}, \text{ when } n_1 < z \leq (1 + D_2) n_1. \end{array}$ Here the upper bound on z is to make sure that the number of winners is still n. Notice that $f(z \sqcup u_{-1})$ is a function that is strately increasing when z increases in such maps, and

 $\int_{u_{1}}^{u_{1}} \left(f_{1}(v \sqcup v_{-1}) - f_{2}(v_{1} \sqcup v_{-1}) \right) \, \mathrm{d} x \geq \int_{u_{1}}^{u_{2}(v)} (\beta(v \sqcup v_{-1}) - f_{2}(v_{1} \sqcup v_{-1})) \, \mathrm{d} x > 0 \ ,$

is control. (2) Because $d^{(0)}$ is 5-DM according to lion 0, in order to prove that $f^{(0)}$ is 5-good, we only need to show that Equation 2.1 holds. As usual, W.L.O.G. we assume $n_1 \ge n_2 \ge \cdots \ge n_n$. We first observe that: $\sum_{i=1}^n f_i^{(0)}(z) a_i = \sum_{i=1}^{n^*} f_i^{(0)}(z) a_i = \frac{1}{n} \cdot \frac{n+D_I}{n^*+D_I} \cdot \sum_{i=1}^{n^*} \frac{a_i(n^*+D_I) - \sum_{i=1}^{n^*} s_i}{D_I}$

 $= \frac{1}{n} \cdot \frac{n + D_{\delta}}{n^* + D_{\delta}} \cdot \left(\sum_{i=1}^{n^*} \tau_i\right).$ For each loser i (i.e., with $i > a^*$), we know that

 $\sum_{i=1}^{n} f_{i}^{(0)}(z)z_{2} + D_{\delta} \cdot f_{i}^{(0)}(z)z_{i} = \sum_{i=1}^{n} f_{i}^{(0)}(z)z_{i} = \frac{1}{n} \cdot \frac{n + D_{\delta}}{n^{i} + D_{\delta}} \cdot \left(\sum_{i=1}^{n^{i}} z_{i}\right) \geq \frac{1}{n} \cdot z_{i} \cdot (n + D_{\delta})$ provides the set of t

 $\sum_{k=0}^{n} f_{1}^{(B)}(z) z_{k} + D_{b} \cdot f_{k}^{(B)}(z) z_{k} = \frac{1}{n} \cdot \frac{n + D_{b}}{n^{2} + D_{b}} \cdot \left(\sum_{k=0}^{n^{2}} z_{k}\right) + D_{b} \cdot f_{k}^{(B)}(z) z_{k}$

 $-\frac{1}{n} \cdot \frac{n + D_4}{u^* + D_4} z_i (n^* + D_5) - \frac{1}{n} \cdot z_i (n + D_5)$
$$\label{eq:response} \begin{split} & n - w + D_{\delta} \sin w + E g (m + E g) = \frac{1}{n} - i \eta (m + E g) \ , \end{split}$$
 again as desired. Notice that this is a generic argument for arbitrary $z = (b, B)^{k}$, and restricting $z = v \in \Sigma = \{0, 1, \ldots, B\}^{k}$, everything still holds; this insidue the proof the is k speed.

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Barlog defined dust given some similar for low care dust or $H^{(0)}_{-1}$ are non-targe to the study of proving that if is shown of H are gravity match is shown of H are gravity match is shown of H are gravity and the second. However, the distance is the distance of the matching the similar properties about $f^{(0)}$ will require some work. For example, result sufficient of the property of matching the similar of ones, because are a piper's hill sufficient of the distance time of the $f^{(0)}$ such that is not observe that the property of matching the similar the expression for $f^{(0)}$, such that is the piper's polarity of winners for distance the $f^{(0)}$ such that is the piper's polarity of winners of one to a distance that the piper's polarity of winners of one to a distance that the piper's polarity of winners of the size of the distance of the size of the distance of the

Lemma E.3. For f⁽⁰⁾ from Definition E.2 the following properties field: 1. The number of winners u^* is well defined. 3. For very $v = (z_1, ..., z_m) \in P, H)^{\frac{N}{2}}$ such that $v_1 \ge v_2 \ge \cdots \ge v_m$ in addition to Equation E.6, we disclose that.

 $\forall i \le u^{*}, -z_{i} > \frac{\sum_{j=1}^{q^{*}} z_{j}}{u^{*} + D_{i}}$, (E.8)

(And an analogous property helds for other z so $T^{(0)}$ is created segment-toolly, J T. The searcher of values v is unique. The searcher of values v is unique. The function $T^{(0)}$ is unique and values T. The function $T^{(0)}$ is unique and values T. T. The function $T^{(0)}$ is d-and T and T and T and T. T. The function $T^{(0)}$ is d-and T and T and T. T. The function $T^{(0)}$ is d-and T and T and T. T. The function $T^{(0)}$ is d-and T and T and T. T. It is a set of T is d-and T and T. T. It is a set of T is d-and T and T. T and T. T is T and T and

Proof. We prove the statements one at a time:

(1) The requirement of Equation E.4 is always satisfied when $n^4 = n$, so the set of possible n^4 's is not empty, and thus the smallest element of that set always exists

(2) As z is assumed to be non-increasing, it suffices to prove Equation E.8 for $i = n^*$. And, indied, by the minimality of n^* we know that if we attempt to choose $n^* - 1$, there exists some $i \ge n^*$ such that

 $u_{t^*} \ge v_f > \frac{\sum_{j=1}^{t^*-1} x_f}{u^* - 1 + D_F}$.

 $= -1 \pm 04$ which after maranging, is equivalent to $z_0 \approx 2\frac{(-1)^2}{2^2 (2^2)}$. Clearly, an analogous statement holds for other z_1 as $f^{(0)}$ is extended symmetrically, by re-labeling the incluse of z to make it non-increasing.

measurements (3) suppose by way of constantistical that there exist two α^* and α^* with $1 \le \alpha^* \le \alpha^* \le \alpha$ ratifying Equation 1.5, in particular, we have advardy statistical that they also satisfy forganize 18.5. Nor datas $\alpha^{-1} = \frac{1}{\alpha_1, \dots, \alpha_k} + \frac{1}{\alpha_k} \frac{1$

 $\frac{S^{*}}{n^{*}+D_{\delta}} \geq n > \frac{S^{*}}{n^{*}+D_{\delta}} = \frac{S^{*}+S^{*}}{n^{*}+n^{*}+D_{\delta}}$

aging over all s_i for $i \in \{n^{\perp} + 1, \dots, n^{\top}\}$, we get $\frac{S^+}{n^++D_3} \gtrsim \frac{S^+}{n^+} > \frac{S^++S^+}{n^++m^++D_2}$

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Finally, we put all the pieces together together. The desired result will follow almost immediately b invoking Diatizguishable Monotonicity Lamma, because it smarres that the undominated strategis are a subset of the possible valuation profiles.

Claim E.4. For every $\delta \in (0, 1)$, and consider the $f^{(0)}$ -machinism $M_{f^{(0)}} = (\Sigma, F_{f^{(0)}})$. For every δ -approximate-valuation profile $K = (K_1, ..., K_n)$, every $\mathbf{t} \in K$, and every $v \in \mathrm{Ubal}(K)$,

 $\mathbb{E}\Big[\mathrm{SW}[\mathsf{tr}, F_{f^{(i)}}(\mathsf{v})) \Big] \geq \frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \cdot \mathrm{MSW}(\mathsf{tr}) \ .$ (8.13)

Proof. By Lemma E.3, the function $f^{(0)}$ from Definition E.2 is a (well-defined) winning-probabilit function that is also 4-good. Therefore, by invoking Lemma E.1 with $f^{(0)}$, we deduce that th $f^{(0)}$ -mechanian $M_{F^{(0)}}$ yields the target guarantee on accial welfare in automizated strategies. Finally, we remark that our proposed optimal probabilistic mechanism can indeed be comp efficiently (not like the second-ortice mechanism):

Lemma E.5. The enforme function $F_{\mu\nu}$ of the $t^{(3)}$ -mechanism $M_{\mu\nu}$ is efficiently computable.

Proof. It suffices to show that both $F_{free}^{(4)} = f^{(6)}$ (the winning-probability function of the mech-mism) and $F_{free}^{(6)}$ (the expected price function of the mechanism) are efficiently computable over

minuto in an $F_{\mu\nu}^{\mu\nu}$ (the expected prior function if the mechanism) are efficiently comparation over $[0,1,\dots,0]^{D-1}$. If $[0,1,\dots,0]^{D-1}$, the mechanism is the minutogravitation prior of γ^{D} is subscriptional difference in comparation, because the mather of existences in the distance in comparation, because the method of the minute of the minute in a short function in a short function of an efficience in the distance in the distance in the minute of the minute of the minute in the minute of the minute in the distance in the minute of the minute in the distance in the minute of the m

$f_n^{(B)}(v_{-n}\cup v_n)\cdot v_n = \int_{v_n}^{v_n} f_n^{(D)}(v_{-n}\cup s) \,\mathrm{d}s$.

Indext, note that $\beta_{n}^{(D)}$ is a function that is preverinely defined according to different solution of ω_{n} mixed in different moders of solutions (i.e., which is the second second

 $-_{\rm ent}g_{\rm eff}$, ore on again see from the proof of the monotonicity of $f^{(0)}$ that the solution of many larger α is non-increasing as a function of x_{α} . Then from, $f_{\alpha}^{(0)}$ contains as in and a difference piece and (in case) piece and in freed, $f_{\alpha}^{(0)}=(x_{\alpha}^{(1)},(x_{\alpha}^{(1)}),(x_{$ When $u_k > \frac{\sum_{i=1}^{N} u_i}{w_{2D_k}}$, one can again see from the proof of the monotonicity of $f^{(0)}$ that the

This is again not hard, by using a simple line sweep method. One can start from $u_n = \frac{\sum_{j=1}^{n-1} D_{j}}{2 + D_{j}}$ and more u_n upwards. At any moment, one can satisfies the earliest time that Equation R is violated, and this that another piece of $A_{j}^{(2)}$ is found.

"1m" view:

animpted with a total probability mass of 1, by averaging three exists none place j such that $P_{0}^{(1)}(\ldots,c_{i}^{(1)}) \leq \frac{1}{2}$, i.e., place preserves the good with as more than $\frac{1}{2}$ redshifty. Without loss $D_{0}^{(1)}(\ldots,c_{i}^{(2)}) \leq \frac{1}{2}$. According to the most of the balance distribution of the second state of the second stat $\vec{e}_{-1} = 0$ consider a "world" with 4-approximate-valuation profile K and true-valuation profile to as

 $K \stackrel{\mathrm{def}}{=} (\mathrm{int}_{\delta}(x), \mathrm{int}_{\delta}(y), \ldots, \mathrm{int}_{\delta}(y))$ and $\operatorname{tr} \stackrel{\mathrm{def}}{=} \left(\lfloor (1+\delta) x \rfloor, \lceil (1-\delta) y \rceil, \dots, \lceil ((1-\delta) y \rceil) \right)$ We have just shown that there exist some $s_1 \cup s'_{-1} \in UDad(K)$ satisfying $F_1^A(s_1 \cup s'_{-1}) = 0$.

 $SW(tv, F(s_1 \cup s_{-1})) = |(1 - \delta)g| \le (1 - \delta)g + 1 \le \frac{x(1 - \delta)^2}{1 + \delta} + 3$ $\leq \frac{(1-\delta)^2}{(1+\delta)^2} \lfloor (1+\delta)x \rfloor + 4 \leq \left(\frac{(1-\delta)^2}{(1+\delta)^2} + \frac{4}{B} \right) \lfloor (1+\delta)x \rfloor$ $= \left(\frac{(1-\delta)^2}{(1+\delta)^2} + \frac{4}{B} \right) MSW(nr) .$ and thus the claimed inequality hods.

E Proof for Theorem 3

E.1 Proof of Statement 1

In 1 result of sourcements in the solution of restrict superbrase to consider only f-mechanism (result Solution 3.2), as no between or binding-induced bornearing frames (format B d) and (f) and (f)

Step 1

Step 1 we start by partiag forward s set of constraints for the winning-pointality function), and prove the they are utilized for ensuing the isage function of the maximum social withins, assuming (for only that the set of available attraction controls with the set of all possible valuation parallels. For easy $\delta \in (0,1)$, doin $m_{1}^{-1} \frac{1}{2} \left[\frac{1}{2} \right]^{2} - 1 = \frac{1}{2} \frac{1}{m_{1}^{2} > 0} > 0$. We age that a similar probability function $\left[i \in A_{2} \right]$ is $i \in A_{2}$ injustly monotonic on δ .

 $\forall i \in N, \forall v \in \{0, 1, ..., B\}^{N}, \sum_{i=1}^{2} f_{j}(v)v_{j} + D_{k} \cdot f_{i}(v)v_{i} \ge \frac{1}{n} \cdot v_{i}(v + D_{k})$. (R.1) We prove the following lemma:

The second inequality of this last equation yields a contradiction with the first inequality in the arms sensition $\frac{S^{\pm}}{n^{\pm}} > \frac{S^{\pm} + S^{\pm}}{n^{\pm} + n^{\pm} + D_4} \Leftrightarrow \frac{S^{\pm}}{n^{\pm}} > \frac{S^{\pm}}{(n^{\pm} + D_4)} \ .$

(1) Substituting Equation E.8 into the definition of p^(B) (defined in Equation E.7) we immediately have p^(B)(z) ≥ 0 for each player i. Summing the p^(B)_i ap over all the players i we get:

 $\sum_{i=1}^{0} f_{i}^{(0)}(z) = \frac{1}{n} \cdot \frac{n+D_{k}}{n^{*}+D_{k}} \cdot \sum_{i=1}^{n^{*}} \frac{z_{i}(n^{*}+D_{k}) - \sum_{j=1}^{n^{*}} z_{j}}{z_{i}D_{k}}$ $= \frac{1}{n} \cdot \frac{n + D_k}{(n^* + D_k)D_k} \cdot \left(n^*(n^* + D_k) - \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} \frac{z_j}{z_i}\right)$

 $\leq \frac{1}{n} \cdot \frac{n + D_4}{(n^* + D_4)D_4} \cdot (n^*(n^* + D_4) - n^*n^*) = \frac{n + D_4}{n} \cdot \frac{n^*}{n^* + D_4} \leq 1 \ .$ In particular, $f_1^{(0)}(z) \leq 1$ for each player i, as we have already established that $f_2^{(0)}(z)$ in mon-negative. And thus, after symmetrically extending the above argument to all other z, we deduce that $f^{(0)}$ is a valid winning-probability function.

(b) For notational simplicity assume that i = n (so that $z_{-1} = z_{-1}$) and also assume $z_2 \ge z_2 \ge \cdots \ge z_{n-1}$. (As usual, the other cases follow by symmetry by appropriate re-infoling.)

So define n' to be number of Winners when only considering the first (n-1) values, i.e., when only considering $z_{0,\dots,n}, z_{n-1}$. We claim that:

 $s_k \leq \frac{\sum_{i=1}^{d} s_i}{d + D_k} \longrightarrow f_0^{(2)}(s) = 0 \text{ (i.e., s is a loser)}$ (E.9) $z_6 > \frac{\sum_{i=1}^{d} z_i}{z_i^2 + D_4} \longrightarrow f_n^{(2)}(z) > 0 \text{ (i.e., κ is a winner)}$ (E.10)

The implication of Equation E.9 is dear, because the number of winners when places a is present is still u'. We now argue that the implication of Equation E 10 also holds which is han of dwidow So assume by way of contradiction that player n is a loser and yet $z_n > \frac{V_n^{(1)} + V_n}{2 + 2 N_n}$. Let the current number of winners be n^{*} (i.e., when player n is present), so we know that $v_m \leq \frac{2\pi m^2}{m^2} \frac{|u|}{|u|}$, in particular, we must have that $u' \neq u^*$ (otherwise we are done, as there is already a contradiction). However, for this choice of n^* , we have that:

 $\forall i \in \{n^* + 1, ..., n - 1\}, \quad z_i \leq \frac{\sum_{j=0}^{n^*} z_j}{n^* + D_2}$ But, this means that both n' and n' are the number of winners when player n is absent, contradiction the universes from item 3. Thus, Ecuation E.10 also holds.

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Lemma E.1. If f is degoed, then the vectorison $M_f = (\Sigma, F_f)$ parameters by ionology the Dirito graduatic Monotonicity Lemma: (Lemma (Lemma E.4) with f has the following serial influer quantum is indeministed statistics: for every h-approximate voltation profile $K = (K_{11}, ..., K_{2n})$, every to π K and every $\pi \in Obs(K_1)$:

$\mathbb{E}\left[SW(tv, F_f(v))\right] \ge \left(\frac{(1-\delta)^2 + \frac{\delta^2}{4}}{(1+\delta)^2}\right) MSW(tv)$

every player $i \in N$, let $x_i \in \mathbb{R}$ be such that $K_i \subset int_2(x_i)$, and let $int_2(x) - int_2(x_3)$.

 $\sum_{i=1}^{n} \operatorname{tr}_{i} f_{j}(v) = \operatorname{SW}(\operatorname{tr}, F_{\ell}(v)) \geq \left(\frac{(1-\ell)^{2} + \frac{4\delta}{2}}{(1+\ell)^{2}} \right) \operatorname{tr}_{i} \; .$ (R.2)

 $\begin{array}{l} \sum_{i=1}^{j \neq i} (i + i + i - j) \\ Alleo, ince every for a large <math display="inline">c \in S^*$ by Lemmas B-A, we must have $(1 - \delta)x_i \leq v_i \leq \max K_i \leq (1 + \delta)x_i, \\ \delta |x_i, \operatorname{sime} f + i + |\partial M| \ and \ K_i \subset \operatorname{int}_{\delta}(x_i). \\ Moreover, \ iv \in K \ indicater \ that, \ (1 - \delta)x_i \leq \Theta_i \leq (1 + \delta)x_i, \\ \operatorname{conthining} \text{these two we have} \end{array}$

 $\begin{array}{l} \label{eq:product} \left\{ h_{1}^{2}, h_{2}^{2}, h_{3}^{2}, h_{4}^{2}, h_{3}^{2}, h_{$

 $\forall i \in \mathbb{N}, \forall v \in \{0, 1, \dots, B\}^{R}, \quad \begin{cases} \sum_{j=1}^{m} (\frac{1-2j}{1+2j})v_j f_j(t) \geq \left(\frac{(1-2j-1)!}{(1+2j)}\right) (\frac{1-2j}{1+2j})v_j f_j(v) > \left(\frac{2j-2j+2!}{(1+2j)}\right) (\frac{1-2j}{1+2j})v_j f_j(v) > \left(\frac{2j-2j+2!}{(1+2j)}\right) (\frac{1+2j}{1+2j})v_j f_j(v) > \left(\frac{2j-2j+2!}{(1+2j)}\right) (\frac{1+2j}{1+2j})v_j f_j(v) > (\frac{2j-2j}{1+2j}) (\frac{2j-2j}{1+2j})$

 $|\psi_{I_1} \psi_{I_2}| = \begin{cases} \sum_{j=0}^{n} \psi_j f_j(v) \ge \left(\frac{(1-\delta)^2 + \delta^2}{(1-\delta)^2}\right) v_i = \frac{4 + D_F}{n} \frac{1}{D^{-1}} - v_i , \\ \sum_{j=0}^{n} \psi_j f_j(v) = D_F v_i f_i(v) \ge \left((1-\delta)^2 + \frac{\delta^2}{n}\right) - \frac{1}{(1-\delta)^2} - v_i = \frac{\pi i D_F}{n} u_i \end{cases}$ (E.4) Note that Equation E.4 is the inequality required by the statement of the lemma, the other in-equality, Equation E.4, we show is implied by Equation E.5. Indiced.

 $\sum_{n=0}^{n} v_2 f_2(v) = \frac{1}{1 + D_2} \left(\sum_{n=0}^{n} v_2 f_2(v) + D_2 v_2 f_1(v) \right) \ge \frac{1}{1 + D_2} \frac{n + D_2}{n} v_1.$ is chained. In sum, we have proved that as long as Equation E.5 is estimized for all $i \in N$ and for any $v \in \{0, 1, ..., R\}^{R}$, the excial welface guarantees of Equation E.2 hold. 21

due, we only need to prove that given two $z_n^{\perp}, z_n^{\perp} \in [0, B]^N$ such that $z_n^{\perp} > z_n^{\perp} >$ $\frac{\sum_{i=1}^{n} z_{i}}{\sum_{i=1}^{n} z_{i}} \text{ we have that } f_{i}^{(0)}(x_{i} \cup z_{i}^{-}) \geq f_{i}^{(0)}(x_{i} \cup z_{i}^{-}). Assume that when bicking z_{i}$ there are a total of $u^{i} = 1$ winners (the first n^{i} players and player n_{i} and bidling z_{i} there are a total of $u^{i} + 1$ winners (the first n^{i} players and player n_{i} . Then we chin that Assume by way of contradiction that $u^{\pm} < u^{\mp} - u^{\pm} + u^{\pm}$. As before, let $S^{\pm} = \sum_{j=1}^{n^{\pm}} z_j$ and $S^{\rm T} = \sum_{i=1}^{N^{\rm T}} v_i - S^{\pm} + S^{\pm}.$ For every player $n^{\pm} \leq i < n$ $\frac{S^{\perp} + z_{0}^{\perp}}{s^{\perp} + 1 + D_{0}} < z_{0} \leq \frac{S^{\top} + z_{0}^{\top}}{s^{\top} + 1 + D_{0}} - \frac{S^{\perp} + S^{\perp} + z_{0}^{\top}}{s^{\perp} + s^{\perp} + 1 + D_{0}}$ Averaging over all $n^{\perp} \leq i < n^{\top}$ we get: $-\frac{S^\lambda+z_0^\lambda}{u^\lambda+1+D_2}<\frac{S^\lambda}{u^\lambda}\leq\frac{S^\lambda+S^\lambda+z_0^\top}{u^\lambda+u^\lambda+1+D_2}$ but this is already a contradiction: $\frac{S^n}{n^n} \leq \frac{S^n + S^n + z_n^{\dagger}}{n^n + n^n + 1 + D_S} \Leftrightarrow \frac{S^n}{n^n} \leq \frac{S^n + z_n^{\dagger}}{n^n + 1 + D_S}$ $= \frac{S^{\pm}}{n^{\pm}} \le \frac{S^{\pm} + z_n^{\pm}}{n^{\pm} + 1 + D_1}$ and thus Equation E.11 holds. nor some Spatient is i i 10000. Nor we show entablished that $u^2 = u^2$. If $u^2 = u^2$ that $\int_0^{R_0} (x_{-u} \cup u_u^2) \geq \int_0^{R_0} (x_{-u} \cup u_u^2)$ is also introduced inspired because they are using the same formula, and send that Equation 12. Is an increasing with respect to z_1 . If $u^2 > u^2$, i.i.th $u^2 = u^2 + u^2$, $u^2 = u^2$, $u^2 = u^2 = u^2 + u^2$, $u^2 = u^2 + u^2$. If $u^2 > u^2 + u^2 = u^2 + u^2$, $u^2 = u^2 + u^2$, and $u^2 = \sum_{i=1}^{n-1} i_i = u^2 + u^2$. By the observations of the $i_i > u^2 + u^2$ is a before. This we average out all $i_i > k < u^2 \le u^2$ we get: $\frac{S^4}{u^4} > \frac{S^4 + z_a^2}{u^4 + 1 + D_4} = \frac{S^7 + S^4 + z_a^4}{u^7 + u^4 + 1 + D_4} \Rightarrow \frac{S^4}{u^6} > \frac{S^7 + z_a^4}{u^7 + 1 + D_4}$ (E.12)

Joing that, we do the calculating:
$$\begin{split} & f_0^{(0)}(x_{-6}+z_0^{-1}) - f_0^{(0)}(x_{-6}+z_0^{-1}) \\ & = C_1 ~ \Big(\frac{z_0^{-1}(w^++1+D_2) - S^{'}-z_0^{-1}}{(w^++1+D_2)z_0^{-1}} - \frac{z_0^{-1}(w^++1+D_2) - S^{*}-z_0^{-2}}{(w^++1+D_2)z_0^{-1}} \Big) \\ & = C_1 ~ \Big(\frac{S^{'}}{(w^++1+D_2)z_0^{-1}} - \frac{S^{''}+z_0^{-1}}{(w^++1+D_2)z_0^{-1}} \Big) \\ & = C_1 ~ \Big(\frac{S^{''}}{(w^++1+D_2)z_0^{-1}} - \frac{S^{''}+z_0^{-1}}{(w^++1+D_2)z_0^{-1}} \Big) \end{split}$$
 $= C_2 \cdot \left(\left(S^+ + z_n^+ \right) (a^+ + 1 + D_2) z_n^+ - \left(S^+ + z_n^+ \right) (a^+ + 1 + D_2) z_n^+ \right)$ $=C_2^{-1}\left((S^{+}+S^{+}+z^{+}_{u})(u^{+}+1+D_{\delta})z^{+}_{u}-(S^{+}+z^{+}_{u})(u^{+}+u^{+}+1+D_{\delta})z^{+}_{u}\right)$ $=O_2^{-1}\left(S^{(2)}(u^{7}+1+D_{4})(z^{7}_{k}-z^{5}_{k})+S^{(5)}(u^{7}+1+D_{4})z^{7}_{k}-u^{8}(S^{(2)}+z^{7}_{k})z^{(6)}_{k}\right)$ $\geq C_{2^{-1}}\left(S^{\top}(u^{\top}+1+D_{\theta})(z_{u}^{2}-z_{u}^{1})+S^{u}(u^{\top}+1+D_{\theta})z_{u}^{2}-u^{u}(S^{\top}+z_{u}^{u})z_{u}^{2}\right)\geq 0$

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See 24 Sec 9 and the d analysis describes the straining periodicity function $f_{\rm c}$ being the Distinguished for the straining of the distinguished for the straining that an archivant will like the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be the support $f_{\rm c}$ and the straining will be straining with the straining will be straining which will be straining with the straining will be straining with the strain

Definition E.2. For every $\delta \in (0, 1)$, and let $D_{\delta} \stackrel{\text{def}}{=} \left(\frac{|\delta|^2}{|\delta|^2} - 1 > 0$. We define the function $\rho(h; |\delta|, B^{(N)} \rightarrow 0, 1|^N)$ are follows:

• for every $z=(z_1,\ldots,z_k)\in[0,B]^N$ such that $z_k\ge z_2\ge\cdots\ge z_k,$ let u^* be the laset index in N such that $\forall i > n^*$, $v_i \le \frac{\sum_{j=0}^{n^*} z_j}{n^* + D_i}$,

call physers $1, \dots, n^n$ the winners and physers $n^n + 1, \dots, n$ the losers, and then set

 $f_{l}^{(0)}(s) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \frac{1}{n} \cdot \frac{n + D_{0}}{n^{2} (\mathcal{B})} \cdot \frac{n (n^{*} + D_{0}) - \sum_{i=1}^{n^{*}} z_{i}}{n D_{i}}, & \text{if } i \leq n^{*}, \\ 0, & \text{if } i > n^{*} ; \end{array} \right.$ (8.7)

• for after z_i define $f^{(0)}$ by retaining it spannetwisely, spacefields, latting r is any permutation more the planers such that $\pi(z) = (z_{i(1)}, \ldots, z_{i(n)})$ is more increasing, we define $f_i^{(0)}(z) \stackrel{\text{def}}{\longrightarrow} f_{i(i)}^{(0)}(z(z))$.

 $\hat{f}^{(0)}_{(0)}(m)$ The detection of $\hat{f}^{(0)}_{(0)}$ are an equivalence of the the multiplying matrix in such that selections. Exact data, the fact is any scening up to the an absoluted proof to $\hat{f}^{(0)}_{(0)}$ is an expansion of the data of the dat

Here we have used the fact of $a_{1}^{(2)}-a_{2}^{(2)}\geq 0$ and $S^{(4)}a^{(1)}+1+B_{0})-a^{(4)}S^{(2)}+a_{2}^{(1)}>0$ (by Equation E.12). This fit induce the proof starting that $f^{(0)}$ is monotonic. The interpolating $f^{(0)}$ is obtained by the constant $B^{(0)}$ in given we estimate a_{1} is a starting of the number of winners decreases when a_{1} increases (woull Equation E.11).

(6) Fix a player i ∈ N and two distinct valuations v₀, v'₀ ∈ (0, 1, ..., B), and assume that v₀ < v'₀. We have already satablished the monotonicity and intergrability of f^(D), so that, to prove that f^(D) is i-DM, we only need to find a specific v₁, to make the integral positive (read)

So define $v_{-1} \stackrel{\text{def}}{=} (v_1, v_2, \dots, v_n)$, then: $\begin{array}{l} f(\eta \cup v_{-1}) = \frac{1}{n} \text{ since these are } n \text{ winners, all bidding the same valuation,} \\ f(v \cup v_{-1}) = \frac{1}{n \log} (D_0 + n - 1 - \frac{n}{n} (n - 1)) > \frac{1}{n}, \text{ when } n_1 < z \leq (1 + D_2) n_1. \end{array}$

Here the upper bound on z is to make sure that the number of winners is still n. Notice that f(z | z = z) is a function that is stratefy increasing when z increases in such manic and $\int_{u_1}^{u_1} \left(f_i(x \cup v_{-i}) - f_i(v_1 \cup v_{-i}) \right) \, \mathrm{d} x \geq \int_{u_1}^{u_1(v_1(\beta + B_i)v_1)} \left(f_i(x \cup v_{-i}) - f_i(v_1 \cup v_{-i}) \right) \, \mathrm{d} x > 0 \ ,$

is consol. (2) Bissman (6) is 3-DM according to lises 0, in order to prove that $f^{(0)}$ is 3-good, we only need to show that Equation 5.1 holds. As usual, W.L.O.G. we assume $\eta_1 \ge \eta_2 \ge \cdots \ge \eta_n$. We first observe that:

 $\sum_{n=1}^{n} f_{i}^{(0)}(z) a_{i} = \sum_{n=1}^{n^{*}} f_{i}^{(0)}(z) a_{i} = \frac{1}{n} \cdot \frac{n+D_{I}}{n^{*}+D_{I}} \cdot \sum_{n=1}^{n^{*}} \frac{a_{i}(n^{*}+D_{I}) - \sum_{i=1}^{n^{*}} s_{i}}{D_{I}}$

 $= \frac{1}{n} \cdot \frac{n + D_2}{n^* + D_3} \cdot \left(\sum_{i=1}^{n^*} \tau_i\right).$ For each loser i (i.e., with $i > n^*$), we know that

 $\sum_{i=1}^{n} f_{i}^{(0)}(z)z_{2} + D_{\delta} \cdot f_{i}^{(0)}(z)z_{i} = \sum_{i=1}^{n} f_{i}^{(0)}(z)z_{i} = \frac{1}{n} \cdot \frac{n + D_{\delta}}{n^{i} + D_{\delta}} \cdot \left(\sum_{i=1}^{n^{i}} z_{i}\right) \geq \frac{1}{n} \cdot z_{i} \cdot (n + D_{\delta})$

provides the set of t

 $\sum_{k=0}^{n} f_{k}^{(B)}(z) v_{k} + D_{b} \cdot f_{k}^{(B)}(z) v_{k} = \frac{1}{n} \cdot \frac{n + D_{b}}{n^{*} + D_{b}} \cdot \left(\sum_{k=0}^{n^{*}} z_{k} \right) + D_{b} \cdot f_{k}^{(B)}(z) v_{k}$

 $-\frac{1}{n} \cdot \frac{n+D_1}{n^*+D_2} z_l(n^*+D_3) - \frac{1}{n} \cdot z_l(n+D_3)$

 $n - w^* + D_0^{-w_1 m_1} - t D_0 r = \frac{1}{n} \cdot n(n + D_0) ,$ again as desired. Notice that this is a generic argument for arbitrary $z \in [0, B]^N$, and where restricting $z = v \in \Sigma = \{0, 1, ..., B\}^N$, everything et ill holds; this finishes the proof that P^N is $\delta = 0.0$.

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Having defined (and given some ministics for) our choice of $f^{(0)}$, we now turn to the task of preving that it antides all the preparities that we must. However, the definition of $f^{(0)}$ is conversion to the task of a statistical statistic

Lemma E.3. For f⁽⁰⁾ from Definition E.2 the following properties field: 1. The number of winners u^* is well defined. 3. For very $v = (z_1, ..., z_m) \in P, W^{(0)}$ such that $v_1 \ge v_2 \ge \cdots \ge v_m$ in addition to Equation E.6, we disclose that.

 $\forall i \le u^*, \quad s_i > \frac{\sum_{j=1}^{n^*} z_j}{u^* + D_i}$. (E.8)

(And we analogous properly holds for effort $x = 0^{-1} + 0.5$ (And we analogous properly holds for effort $x = 0^{-10}$ is actualed symmetrically,) 3: The number of simulation is a simulation of the second statements 3: The function 1^{-0} is unadvance and statements. 5: The function 1^{-0} is $1 \cdot defortgapsilolity ensembles:$ $5: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $5: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $5: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$ $6: The function <math>1^{-0}$ is $1 \cdot defortgapsilolity ensembles:$

Proof. We prove the statements one at a time:

(1) The requirement of Equation E.4 is always satisfied when $n^4 = n$, so the set of possible n^4 's is not empty, and thus the smallest element of that set always exists

(2) As z is assumed to be non-increasing, it suffices to prove Equation E.8 for i = n^{*}. And, indeed, by the minimality of n^{*} we know that if we attempt to choose n^{*} = 1, there exists some $i \ge n^*$ such that

 $z_{0^*} \ge z_j > \frac{\sum_{j=0}^{N^*-1} z_j}{u^* - 1 + D_2}$,

 $= -1 \pm 04$ which after maranging, is equivalent to $z_0 \approx 2\frac{(-1)^2}{2^2 (2^2)}$. Clearly, an analogous statement holds for other z_1 as $f^{(0)}$ is extended symmetrically, by re-labeling the incluse of z to make it non-increasing.

momentum matrix and the state of the state

 $\frac{S^{*}}{n^{*}+D_{\delta}} \geq n > \frac{S^{*}}{n^{*}+D_{\delta}} = \frac{S^{*}+S^{*}}{n^{*}+n^{*}+D_{\delta}}$

aging over all s_i for $i \in \{n^{\perp} + 1, \dots, n^{\top}\}$, we get $\frac{S^{\pm}}{n^2+D_3} \gtrsim \frac{S^{\pm}}{n^4} > \frac{S^{\pm}+S^{\pm}}{n^4+m^4+D_2}$

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Finally, we put all the pieces together together. The desired result will follow almost immediately b invoking Diatizguishable Monotonicity Lamma, because it smarres that the undominated strategis are a subset of the possible valuation profiles.

Claim E.4. For every $\delta \in (0, 1)$, and consider the $f^{(0)}$ -machinism $M_{f^{(0)}} = (\Sigma, F_{f^{(0)}})$. For every δ -approximate-valuation profile $K = (K_1, ..., K_n)$, every $\mathbf{t} \in K$, and every $v \in \mathrm{Ubal}(K)$,

 $\mathbb{E}\Big[\mathrm{SW}(\mathrm{tr},F_{f(0)}(\mathbf{v}))\Big] \geq \frac{(1-\delta)^2+\frac{4\delta}{v}}{(1+\delta)^2} \ \ \mathrm{MSW}(\mathrm{tr}) \ .$ (15.12)

Proof. By Lemma E.3, the function $f^{(0)}$ from Definition E.2 is a (well-defined) winning-probabilit function that is also 4-good. Threefers, by invoking Lemma E.1 with $f^{(0)}$, we deduce that th $f^{(0)}$ -mechanism $M_{F^{(0)}}$ yields the target guarantee on social welfare in unionizated strategies.

Pinally, we remark that our proposed optimal probabilistic mechanism can indeed be a efficiently (part like the second-price mechanism)

Lemma E.5. The enforme function $F_{\mu\nu}$ of the $t^{(3)}$ -mechanism $M_{\mu\nu}$ is efficiently computable. Proof. It suffices to show that both $F_{free}^{(4)} = f^{(6)}$ (the winning-probability function of the mech-mism) and $F_{free}^{(6)}$ (the expected price function of the mechanism) are efficiently computable over

 $[0, 1, ..., \theta]^N$. First, we note that the winning-probability function $f^{(0)}$ is indeed efficiently computable, be First, we note that the winning-probability function $f^{(0)}$ is indeed efficiently computable, be This is the data we summit preventing matrices μ is stated ending comparison. Since the number of values is below 1 and a and can be determined in linear time. Next, we argue why the expected pixet function is also efficiently computable, which may not be colorison so it is defined an integral of $f^{(0)}$ (see Definition B.2). So, without loss of generality we show how to compute the expected pixet pixet player n as a function of u_{11} , n_{22} .

 $f_n^{(B)}(v_{-n}\cup v_n) \cdot v_n = \int$

ndeed, note that $f_n^{(D)}$ is a func-rent different values of - - and different different values of assume that $v_1 \ge v_2 \ge \cdots \ge$ When $v_0 \le \frac{\sum_{n=1}^{\infty} (v_1)}{n! (20)}$, the that $f_n^{(b)} = 0$, so that integr When $n_{\rm H} > \frac{\sum_{i=1}^{n} v_i}{n^2 + 10^2}$ This is again and move up ups
Approximate Valuations

